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A Saturation Property Concerning a General Probabilistic Representation of Operator Semigroups

FANG YOU-JIAN

Department of Mathematics, Hubei University, Wuhan, People's Republic of China

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I. INTRODUCTION

Let X be a real random variable on some probability space $(\Omega, \mathscr{A}, P)$. The symbols $E(X)$, $\sigma^2(X)$, $\psi_X(t) = E(t^X)$, and $\psi_X^*(t) = E(e^{tX})$ denote the expected value, the variance, the probability generating function, and the moment-generating function of X, respectively. For a real Banach space $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ with adjoint space $(\mathfrak{X}^*, \|\cdot\|_{\mathfrak{X}^*})$, let $\mathscr{E}(\mathfrak{X})$ denote the Banach algebra of bounded endomorphisms on $\mathfrak{X}$ and let $\{T(t); t \geqslant 0\}$ denote a strongly continuous operator semigroup with infinitesimal generator A. As usual, $D(A^k)$ $(k = 1, 2, ...)$ denotes the domain of A^k. We know that there are constants $M \geqslant 1$ and $\omega \geqslant 0$ such tht

$$\|T(t)\| \leqslant Me^{\omega t} \qquad (t \geqslant 0), \tag{1}$$

where $\|\cdot\|$ is the operator norm in $\mathscr{E}(\mathfrak{X})$ (for details on operator semigroups, see, for instance, [1]).

In this paper, we consider a general probabilistic representation of operator semigroups given by D. Pfeifer [6, 7] which extends a similar result due to Chung [2, Th. 5] and is stated in [7] as follows.

COROLLARY 4. *Let N be a non-negative integer-valued random variable with $E(N) = \zeta$ and let $Y \geqslant 0$ be a real-valued random variable with $E(Y) = \gamma$ such that $\psi_N(\delta_1) < \infty$ for some $\delta_1 > 1$ and $\psi_Y^*(\delta_2) < \infty$ for some $\delta_2 > 0$. Then for sufficiently large $n \in \mathbb{N}$, $\psi_N(E[T(Y/n)]) \in \mathscr{E}(\mathfrak{X})$ with*

$$\|\psi_N(E[T(Y/n)])\| \leqslant M\psi_N(\psi_Y^*(\omega/n))$$

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and

$$T(\xi)=\lim_{n\to\infty}\{\psi_N(E[T(Y/n)])\}^n \tag{2}$$

in the strong sense where $\xi=\zeta\gamma$.

In the above corollary, $\mathbb{N}$ is the set of natural numbers and $E(T(Y/n))$ is defined by means of an extended Pettis integral [7].

We consider representation formula (2) because almost all known representation formulas of operator semigroups can be obtained from it by specialization. The convergence rate of representation (2) has been studied in [8]. Here, we present its saturation property.

THEOREM A. *Suppose* $I_1, I_2 \subseteq (0,\infty)$ *are two intervals. Let* N_η, $\eta\in I_1$, *be a non-negative integer-valued stochastic process with* $E(N_\eta)=\eta$ *and let* $X_\lambda\geqslant 0$, $\lambda\in I_2$, *be a real-valued stochastic process with* $E(X_\lambda)=\lambda$ *such that* (i) $\sigma^2(X_\lambda)$, $\sigma^2(N_\eta)<\infty$ *and* $\eta\sigma^2(X_\lambda)+\lambda^2\sigma^2(N_\eta)>0$ *for* $\lambda\eta>0$. (ii) $\psi^*_{X_\lambda}(\delta)<\infty$, $\psi_{N_\eta}(\psi^*_{X_\lambda}(\delta))<\infty$ *for some* $\delta>0$, *and* $\eta\leqslant a$, *for some* $a>0$. (iii) *For* $b>0$, $(0,b]\subseteq\{\xi=\lambda\eta;\ \eta\in I_1,\ \lambda\in I_2\}\subseteq[0,\infty)$ *the following are equivalent.*

(1) *For* $\forall f\in D(A^k)$ $(k\in\mathbb{N})$ *and* $\forall\xi\in(0,b)$,

$$\|T(\xi)f-\{\psi_{N_\eta}(E[T(X/n)])\}^n f\|_{\mathfrak{X}}=o_{\eta,\lambda}(1/n)\qquad(n\to\infty).$$

(2) $T(t)=I+Bt$, *where* $t\in[0,\infty)$, B *is a bounded linear operator with* $B^2=0$, *and* I *is the identity operator.*

Theorem A tells us the interesting fact that the local approximation property (1) (b can be very small) implies the global property (2) of the operator semigroup.

We point out that the bounded linear operator B in (2) of Theorem A can be non-trivial. For example, $\mathscr{B}=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right);\ a,b,c,d\in R\}$ is a Banach space with norm $\|\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\|=(a^2+b^2+c^2+d^2)^{1/2}$. For $X=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathscr{B}$, put $Bx=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$, $IX=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. Then B is a bounded linear operator on $\mathscr{B}$ with $B\neq 0$ and $B^2=0$, and I is the identity operator on $\mathscr{B}$. We can see that $\{T(t)=I+Bt;\ t\geqslant 0\}$ is a strongly continuous semigroup of operators.

Let $N_\eta\equiv 1$. We get

COROLLARY B. *Suppose* $X_\lambda\geqslant 0$, $\lambda\in[0,b]$ *or* $(0,b)$, *is a stochastic process such that* (i) $E(X_\lambda)=\lambda$, (ii) $0<\sigma^2(X_\lambda)<\infty$ *for* $\lambda\in(0,b)$, (iii) $\psi^*_{X_\lambda}(\delta)<\infty$ *for some* $\delta>0$. *Then the following are equivalent.*

(a) *For* $\forall f \in D(A^k)$ $(k \in \mathbb{N})$ *and* $\forall \lambda \in (0, b)$,

$$\| T(\lambda) f - \{E[T(X_\lambda/n)]\}^n f \|_{\mathfrak{X}} = o_\lambda(1/n) \qquad (n \to \infty).$$

(b) $T(t) = I + Bt$, *where* $t \in [0, \infty)$, B *is a bounded linear operator with* $B^2 = 0$.

It is easy to show that Theorem A applies to the following representation formulas of operator semigroups (see [6]):

(1) $T(\xi) = \lim_{n \to \infty} \exp(\xi n [T(1/n) - I])$.

(2) $T(\xi) = \lim_{n \to \infty} \exp(-\xi n I + \xi n^2 R(n))$ where $R(n) = \int_0^\infty e^{-nt} T(t)\, dt$.

(3) $T(\xi) = \lim_{n \to \infty} (I - \xi I + \xi T(1/n))^n$.

(4) $T(\xi) = \lim_{n \to \infty} (I - \xi I + \xi n R(n))^n$.

(5) $T(\xi) = \lim_{n \to \infty} (I + \xi I - \xi T(1/n))^{-n}$.

(6) $T(\xi) = \lim_{n \to \infty} (I + \xi I - \xi n R(n))^{-n}$.

(7) $T(\xi) = \lim_{n \to \infty} (2I - T(\xi/n))^{-n}$.

(8) $T(\xi) = \lim_{n \to \infty} ((n/\xi)\, R(n/\xi))^n$.

(9) $T(\xi) = \lim_{n \to \infty} \{(n^\xi / T(\xi)) \int_0^\infty t^{\xi - 1} e^{-nt}\, T(t)\, dt\}^n$.

(10) $T(\xi) = \lim_{n \to \infty} \{(n/2\xi) \int_0^{2\xi/n} T(t)\, dt\}^n$.

II. Lemmas

Let N_η, $\eta \in I_1$, be a non-negative integer-valued stochastic process on probability space $(\dot{\Omega}, \mathscr{A}, \dot{P})$ with $E(N_\eta) = \eta$, and let $X_{1\lambda}, X_{2\lambda}, ..., \lambda \in I_2$, be non-negative real-valued stochastic processes on $(\Omega, \mathscr{A}, P)$ with $E(X_{1\lambda}) = \lambda$ such that for each fixed $\eta \in I_1$ and each fixed $\lambda \in I_2$, $N_\eta, X_{1\lambda}, X_{2\lambda}, ...$ are independent and $X_{1\lambda}, X_{2\lambda}, ...$ are identically distributed. Put $Y_{n\lambda} = (\sum_{k=1}^n X_k)/n$ and $Z_{\lambda,\eta} = \sum_{k=1}^{N_\eta} X_{k\lambda}$. Then the following lemmas are valid. Incidentally, C and C_i $(i = 0, 1, 2, ...)$ used in the following are positive constants independent of n.

Lemma 1. $E|Y_{n\lambda} - \lambda|^{2m} \leqslant C n^{-m} E|X_{1\lambda} - \lambda|^{2m}$ *(see* [5]*)*.

Lemma 2. *If* $\psi^*_{X_{1\lambda}}(\delta) < \infty$ *for some* $\delta > 0$, *then*

$$\psi^*_{Y_{n\lambda}}(\gamma) \leqslant C_0 < \infty \qquad (r > 0, n > r/\delta).$$

Lemma 3. *If* $\psi^*_{X_{1\lambda}}(\delta) < \infty$, $\psi_{N_\eta}(\psi^*_{X_{1\lambda}}(\delta)) < \infty$ *for some* $\delta > 0$ *then*

$$\psi^*_{Z_{\lambda,\eta}}(\delta) = \psi_{N_\eta}(\psi^*_{X_{1\lambda}}(\delta)).$$

LEMMA 4. *If* $\psi^*_{X_{1\lambda}}(\omega) < \infty$, $\psi_{N_\eta}(\psi^*_{X_{1\lambda}}(\omega)) < \infty$, *then*

$$\begin{aligned} E(T(Z_{\lambda,\eta})) &= \psi_{N_\eta}(E[T(X_{1\lambda})]) \\ &= \sum_{m=0}^{\infty} P(N_\eta = m)(E[T(X_{1\lambda})])^m \qquad (\textit{see } [7]). \end{aligned} \tag{3}$$

LEMMA 5. *For a real continuous function* $g(x)$ *on* $[0, \infty)$ *with* $\sup |g(x)| \leqslant C_1 e^{\omega x}$, let

$$E_n(f, \lambda) = \int_\Omega g(Y_{n\lambda})\, dP = \int_0^\infty g(t)\, dF_{Y_{n\lambda}}(t) \qquad (\textit{see } [4, 9]). \tag{4}$$

Suppose that (i) $I_1 \supseteq (0, b]$, (ii) $0 < \sigma^2(X_{1\lambda}) < \infty$ for $\lambda \in (0, b)$, (iii) $\psi^*_{X_{1\lambda}}(\delta) < \infty$ for some $\delta > 0$. Then

$$|E_n(g, \lambda) - g(\lambda)| = o_\lambda(1/n) \qquad (n \to \infty)$$

if and only if

$$g(x) = g(0) + (g(b_1) - g(0)) b_1^{-1} x \qquad (0 \leqslant x \leqslant b, \quad 0 < b_1 < b). \tag{5}$$

Note that Lemma 5 is the o-saturation theorem of the probabilistic operator (4).

The proof of Lemmas 2 and 3 are quite easy. Now we prove Lemma 5.

If $g(x)$ is not a linear function on $[0, b]$ then

$$G(x) = g(x) - g(0) - (g(b) - g(0)) b^{-1} x \not\equiv 0 \qquad (0 \leqslant x \leqslant b).$$

Because $G(0) = G(b) = 0$, it follows that $\exists x_0 \in (0, b)$ such that $G(x_0) \neq 0$, say $G(x_0) > 0$. By the parabolic curve lemma [3], there are $y \in (0, b)$ and a polynomial

$$Q(x) = a(x - y)^2 + c(x - y) + G(y) \qquad (a < 0)$$

such that $Q(x) \geqslant G(x)$ $(x \in [0, b])$, $Q(y) = G(y)$, and $Q(0) > 0$, $Q(b) > 0$. Put $b' = \min\{x; x > b, Q(x) = G(x)\}$. Then $b' = \infty$ or $b' \neq (b, \infty)$. We set

$$h(x) = \begin{cases} 0, & x \in [0, b') \\ G(x) - Q(x), & x \in [b', \infty). \end{cases} \tag{6}$$

Obviously,

$$\begin{aligned} |h(x)| &\leqslant |g(x)| + |g(0) + (g(b) - g(0)) b^{-1} x| + |Q(x)| \\ &\leqslant C_2 e^{\omega x} \end{aligned} \tag{7}$$

and $G(x) \leqslant Q(x)+h(x)$ $(\forall x \geqslant 0)$. Applying the conditions of Lemma 5 and the results obtained above, we have

$$\begin{aligned} E_n(G(x), y)-G(y) &\leqslant E_n(Q(x)+h(x)-G(y), y) \\ &= E_n(a(x-y)^2+c(x-y), y)+E_n(h(x), y) \\ &= a\sigma^2(Y_{n\lambda})+E_n(h(x), y). \end{aligned}$$

It is easy to show $|h(x)| \leqslant C_3 e^{\omega x}(x-y)^4$ by (6) and (7). Hence

$$\begin{aligned} |E_n(h(x), y)| &\leqslant C_3 E_n(e^{\omega x}(x-y)^4, y) \\ &\leqslant C_3(E_n(e^{2\omega x}, y)\, E_n((x-y)^8, y))^{1/2} \\ &= C_3(E(e^{2\omega Y})\, E((Y_{ny}-y)^8))^{1/2}; \end{aligned}$$

therefore, by Lemma 1 and Lemma 2,

$$E_n(G(x), y)-G(y) \leqslant a\sigma^2(X_{1y})/n+o_y(1/n) \qquad (a<0)$$

which is in contradiction with the following fact

$$E_n(G(x), y)-G(y)=E_n(g(x), y)-g(y)=o_y(1/n).$$

III. Proof of Theorem A

First, we prove Corollary B. On the basis of (b), we have

$$\{E[T(X_\lambda/n)]\}^n=\{E(I+BX_\lambda/n)\}^n=\{I+2B/n\}^n=I+B\lambda.$$

So (b) implies (a).

In order to show (a) $\Rightarrow$ (b), without loss of generality, we suppose $X_\lambda=X_{1\lambda}$ in Lemma 5. For each fixed $f^* \in \mathfrak{X}^*$ and each fixed $f \in \mathfrak{X}$,

$$|f^*(T(x)f)| \leqslant \|f^*\|_{\mathfrak{X}^*} \|f\|_{\mathfrak{X}} Me^{\omega x}$$

and by the definition of the extended Pettis integral [7], we have

$$\begin{aligned} E_n(f^*(T(\cdot)f), \lambda) &= \int_\Omega f^*(T(Y_{n\lambda})f)\, dP \\ &= f^*[E(T(Y_{n\lambda}))f] \\ &= f^*[(ET(X_\lambda/n))^n f]. \end{aligned}$$

According to (a),

$$|E_n(f^*(T(\cdot)f),\lambda)-f^*(T(\lambda)f)|=o_\lambda(1/n) \qquad (n\to\infty).$$

By Lemma 5,

$$f^*(T(x)f)=f^*(T(0)f)+[f^*(T(b_1)f) \\ -f^*(T(0)f)]b_1^{-1}x \qquad (0\leqslant x\leqslant b,\quad 0<b_1<b).$$

So,

$$T(x)f=f+(T(b_1)-I)b_1^{-1}fx \qquad (0\leqslant x\leqslant b,\quad 0<b_1<b).$$

Let $b_1\to 0$. We get

$$T(x)f=(I+Ax)f \qquad (0\leqslant x\leqslant b);$$

therefore

$$T(x)=I+Ax \qquad (0\leqslant x\leqslant b),$$

where A is the infinitesimal generator of $\{T(t);t\geqslant 0\}$.

It is easy to see, by the properties of operator semigroups, that A is a bounded linear operator here and that $A^2=0$.

If $x>b$, then there exists an integer $n=[x/b]$ such that $x=nb+\beta$ $(0\leqslant\beta<b)$. Hence

$$T(x)=[T(b)]^n\,T(\beta)=I+Ax.$$

Now we prove Theorem A. Let $N_\eta, X_{1\lambda}, X_{2\lambda}, ..., Z_{\lambda\eta}=\sum_{k=1}^{N_\eta}X_{k\lambda}$ are as in Section II, and let $X_{1\lambda}=X_\lambda$ and N_η satisfy the conditions of Theorem A. Then

(1) $E(Z_{\lambda\eta})=E(\sum_{k=1}^{N_\eta}X_{k\lambda})=\eta\lambda=\xi$;

(2) when $\xi>0$, $\sigma^2(Z_{\lambda\eta})=\eta\sigma^2(X_\lambda)+\lambda^2\sigma^2(N_\eta)>0$;

(3) by Lemma 3, $\psi^*_{Z_\eta^2}(\delta)=\psi_{N_\eta}(\psi^*_{X_\lambda}(\delta))<\infty$;

(4) by Lemma 4, $E(T(Z_{\lambda\eta}/n))=\psi_{N_\eta}(E[T(X_\lambda/n)])$ $(\eta>\omega/\delta)$ and we replace X_λ in Corollary B with $Z_{\lambda\eta}$. Then we can see that Theorem A is valid.

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